Some Two-Mode Buckling Problems and Their Relation to Catastrophe Theory

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The asymptotic buckling analysis of semisymmetric, two-degree-of-freedom static systems is presented. The types of behavior which can occur for this class of system are determined and categorized in accordance with the results of catastrophe theory. The expressions for the critical load-initial imperfection surfaces are determined in closed form. Within the context of an asymptotic analysis all possible secondary bifurcations cases are isolated. (The term bifurcation is used here in the sense which is usual in elastic stability analyses and not in the broader sense which is common in catastrophe theory.) In addition, the physical significance of the different critical load-initial imperfection surfaces is determined, and the primary surface is identified for each case. The general results are demonstrated in the example of the two-mode buckling of an axially loaded beam resting on a nonlinear elastic foundation.

Introduction

MULTIPLE-mode buckling problems are defined by the characteristic that the least eigenvalue of the linearized equations is not unique. As a result of the nonuniqueness, the corresponding eigenfunction is defined by a combination of eigenmodes and thus pre- and postbuckling analyses rely on the solution of simultaneous nonlinear algebraic equations which abounds with difficulties. Multiple-mode problems are, however, extremely important. They are a natural result in many nonlinear shell studies while they have been introduced artificially through some optimization attempts. An example is the optimization of stiffened panels such that the Euler and local buckling loads coincide. As such, multiple-mode buckling of shells and of stiffened structures has been thoroughly investigated. Much effort has been expended to understand the idiosyncrasies of each individual problem. The reader should consult, for example, the review paper by Hutchinson and Koiter,[1] the books by Thompson and Hunt[2] and by Huseyin,[3] or the proceedings of the *IUTAM Symposium on Buckling of Structures*.[4]

On the other hand, the analysis of multiple-mode problems in general terms has not received as much attention, although solid contributions have been made. In particular, for systems containing no initial imperfections the papers by Supple,[5] Chilver,[6] Sewell,[7] and Johns and Chilver[8] may be mentioned. In addition, some of the most powerful and interesting results for general imperfect systems are those due to Ho.[9,10]

The purpose of the present paper is to present an analysis which, with the results of Thom's[11] catastrophe theory in mind, will provide a classification scheme for many two-mode buckling problems. The classification system presented may therefore be viewed as an attempt to categorize simple two-mode systems in a manner which is analogous to the schemes for single-mode systems as given in Ref. 2.

A comparison between Thom's results and asymptotic buckling theory has been recently presented in an article by Thompson and Hunt,[12] and certain of their results are closely related to those presented here. This paper considers the group of problems in which there are two coincident least eigenvalues and in which the highest-order terms in the potential energy are cubics of the so-called semisymmetric form. The significant aspect of this form is that problems of this type have a very close tie with two of Thom's elementary catastrophes, the hyperbolic umbilic and the elliptic umbilic. It is possible to transform an arbitrary cubic potential energy expression into the form of these catastrophes.[13] However, it seems that, in certain physical situations, it may be of some advantage to examine the stability of a system in its natural coordinate system and with its natural control parameters. That is, the deflections, loads, and imperfections retain their individual physical significance. Such a procedure results in a number of subgroups within the classification given by catastrophe theory; however, Thom's work still provides the basis for the overall classification scheme of these two-mode systems.

The first portion of this paper presents the methodology by which the semisymmetric potential energy expression arises in the analysis of elastic structures. The second part is then devoted to a general discussion of the semisymmetric form. In this section the closed-form general expressions for the asymptotic critical load-initial imperfection surfaces as well as the critical load-imperfection curves for all secondary bifurcation possibilities are determined. The final section demonstrates the general asymptotic results. The example chosen is that of an axially loaded beam resting on a nonlinear elastic foundation. Graphical results are presented for all of the subclasses encountered in the analysis.

Semisymmetric Potential Energy

The method of analysis as well as the functional notation follows that developed by Koiter.[14] Thus, the potential energy of the system can be expressed as

$$P^{\lambda}[\psi]=P_1^{\lambda}[\psi]+P_2^{\lambda}[\psi]+P_3^{\lambda}[\psi]+\ .\ .\ .\ +Q_1^{\lambda}[\psi] \tag{1}$$

In Eq. (1), λ is the applied load, ψ is the deflection of the system, and $P^{\lambda}[\psi]$ is the total potential energy which is, in turn, expanded in terms of $P_i^{\lambda}[\psi]$ and $Q_1^{\lambda}[\psi]$. The quantities $P_i^{\lambda}[\psi]$ are functionals of the ith degree in ψ, while $Q_1^{\lambda}[\psi]$ is of the first degree in ψ and contains the influence of the initial imperfection. All higher order terms have been omitted on the grounds that they are of lesser importance. For two-mode systems ψ_1, the eigenfunction corresponding to the least eigenvalue λ_1 of the linearized problem takes the form

$$\psi_1=au+bv \tag{2}$$

where u and v are the eigenmodes, and a and b are the modal amplitudes. Thus, following Koiter,[14] the potential energy of

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the system is expanded about the ideal state, or about the state corresponding to the least eigenvalue to yield a first-order approximation to the potential energy in the form

$$P^{\lambda}[\psi_1] \equiv F(a,b) = (\lambda-\lambda_1)P_2'[\psi_1] + P_3[\psi_1] + Q_1[\psi_1] \quad (3)$$

where

$$P_2'[\psi_1] = a^2P_2'[u] + abP_{11}'[u,v] + b^2P_2'[v] \quad (4)$$

$$P_3[\psi_1] = a^3P_3[u] + a^2bP_{21}[u,v] + ab^2P_{12}[u,v] + b^3P_3[v] \quad (5)$$

$$Q_1[\psi_1) = aQ_1[u] + bQ_1[v] \quad (6)$$

In the foregoing expressions, $P_1[\psi_1]$ does not appear as $P_1[\psi_1] \equiv 0$ by virtue of the equilibrium requirement. Also, a prime indicates differentiation with respect to λ and then evaluation at $\lambda = \lambda_1$, while the lack of a λ superscript indicates evaluation at $\lambda = \lambda_1$. Terms of the form $P_{ij}[u,v]$ are functionals of the ith degree in u and the jth degree in v, and, by definition, $P_{11}'[u,u] = 2P_2'[u]$ and $P_{21}[u,u] = 3P_3[u]$. Furthermore, the orthogonality of the eigenfunctions u and v yields the result that $P_{11}'[u,v] = 0$. The particular situation which is of interest arises when either $P_{21}[u,v] = P_3[v] = 0$ or $P_3[u] = P_{12}[u,v] = 0$, which are evidently equivalent situations. Thus, for the purposes of analysis the first-order approximation to the potential energy takes the form

$$F(a,b) = a^3P_3[u] + ab^2P_{12}[u,v] + (\lambda-\lambda_1)a^2P_2'[u] + (\lambda-\lambda_1)b^2P_2'[v] + aQ_1[u] + bQ_1[v] \quad (7)$$

Since the cubic portion of this expression is symmetric in b this is referred to as the semisymmetric form. It must be cautioned that, if either $P_3[u]$ or $P_{12}[u,v]$ in Eq. (7) vanishes, then higher-order terms must be included in the potential energy expression and what is to follow would not be valid. Such a situation is discussed in greater detail in Ref. 13. In addition, it should be noted that Eq. (7) is precisely the form which was investigated in Ref. 15 and that the energy expression considered in Ref. 16 arises when $P_3[u]$ is missing.

Equilibrium and Stability of the Semisymmetric Form

The equilibrium equations for a system whose potential energy is described by Eq. (7) are obtained by partial differentiation of Eq. (7) with respect to a and b and then setting each of the partial derivatives to zero. Doing so yields

$$2(\lambda-\lambda_1)aP_2'[u] + 3a^2P_3[u] + b^2P_{12}[u,v] = -Q_1[u] \quad (8)$$

and

$$2(\lambda-\lambda_1)bP_2'[v] + 2abP_{12}[u,v] = -Q_1[v] \quad (9)$$

The stability determinant is also readily obtained, and upon evaluation of the determinant the critical states of equilibrium are defined to occur when

$$\{(\lambda-\lambda_1P_2'[u] + 3aP_3[u]\}\{(\lambda-\lambda_1)P_2'[v] + aP_{12}[u,v]\} - b^2\{P_{12}[u,v]\}^2 = 0 \quad (10)$$

Since the critical load is a function of two imperfection parameters, the complete solution is critical load-imperfection surfaces in the three-dimensional λ_c, $Q_1[u]$, $Q_1[v]$ space. These surfaces are obtained through the elimination of a and b from Eqs. (8-10). This operation is not particularly difficult, but it does require some caution in the sequencing of the various steps involved. The final solution takes the form

$$[X \pm Y^{1/2}][-3X \pm Y^{1/2}]^3 = Z \quad (11)$$

with

$$X = (\lambda-\lambda_1)[\alpha_1 - 3\alpha_2] \quad (12a)$$

$$Y = 3(\lambda-\lambda_1)^2[3\alpha_1^2 - 2\alpha_1\alpha_2 + 3\alpha_2^2] - 24\alpha_3Q_1[u] \quad (12b)$$

$$Z = (12)^3\alpha_3\alpha_4\{Q_1[v]\}^2 \quad (12c)$$

where

$$\alpha_1 = P_2'[u]P_{12}[u,v] \quad (13a)$$

$$\alpha_2 = P_2'[v]P_3[u] \quad (13b)$$

$$\alpha_3 = P_3[u]\{P_{12}[u,v]\}^2 \quad (13c)$$

$$\alpha_4 = \{P_3[u]\}^2P_{12}[u,v] \quad (13d)$$

This expression can be transformed to a quartic in $(\lambda-\lambda_1)^2$ and although solutions of the form

$$(\lambda-\lambda_1)^2 = f(Q_1[u], Q_1[v]) \quad (14)$$

are obtainable, they are not straightforward. On the other hand, since it is only the surfaces which are of interest, the solution can easily be expressed as

$$Q_1[v] = \pm[g\{(\lambda-\lambda_1), \quad Q_1[u]\}]^{1/2} \quad (15)$$

since Eq. (11) is a square in $Q_1[v]$. Equation (15) is, of course, not defined for arbitrary values of $(\lambda-\lambda_1)$ and $Q_1[u]$. That is, both Y and $g\{\cdot,\cdot\}$ must be positive or zero.

A particularly interesting feature of the critical load-initial imperfection surfaces are those regions which are associated with secondary bifurcation states. Here secondary bifurcation is taken to mean bifurcation states which result for nontrivial values of a and/or b. The reason for this interest is that vertical slopes of the critical load-initial imperfection surfaces with the implied dramatic changes in behavior always characterize bifurcation states.[17]

The simplest manner to isolate the bifurcation states is to note that the derivative of the load with respect to either of the imperfection components will be infinite at the bifurcation load.[17] Alternatively, the derivative of either of the imperfection parameters with respect to the load will be zero. For this particular problem, it can be shown that this criterion is analogous to specifying that the critical state is not a limit point and is therefore a bifurcation state.

Differentiating Eq. (11) with respect to λ, while maintaining $Q_i[v]$ constant, yields

$$\left[\frac{dX}{d\lambda} \pm \frac{1}{2}Y^{-1/2}\frac{dY}{d\lambda}\right]\left[-3X \pm Y^{1/2}\right]^3 + 3[X \pm Y^{1/2}] \times [-3X \pm Y^{1/2}]^2\left[-3\frac{dX}{d\lambda} \pm \frac{1}{2}Y^{-1/2}\frac{dY}{d\lambda}\right] = 0 \quad (16)$$

where

$$\frac{dX}{d\lambda} = [\alpha_1 - 3\alpha_2], \quad \frac{dY}{d\lambda} = 6(\lambda-\lambda_1)[3\alpha_1^2 - 2\alpha_1\alpha_2 + 3\alpha_2^2] \quad (17)$$

Equation (16) has two possible solutions: either

$$[-3X \pm Y^{1/2}] = 0 \quad (18)$$

or

$$\left[\frac{dX}{d\lambda} \pm \frac{1}{2} Y^{-1/2} \frac{dY}{d\lambda}\right]\left[-3X \pm Y^{1/2}\right] + 3\left[X \pm Y^{1/2}\right]\left[-3\frac{dX}{d\lambda} \pm Y^{-1/2}\frac{dY}{d\lambda}\right] = 0 \tag{19}$$

Consider first the bifurcation state defined by Eq. (18). Transferring $\pm Y^{1/2}$ to the right-hand side of the equation, squaring both sides, and then solving for the bifurcation load yields

$$(\lambda - \lambda_1)^2 = \frac{-\alpha_3 Q_1[u]}{\alpha_2(3\alpha_2 - 2\alpha_1)} \tag{20}$$

This solution is real only if the following conditions are satisfied: 1) the right-hand side of Eq. (20) is positive, 2) $Q_1[v]$ is zero [since $[-3X \pm Y^{1/2}] = 0$ implies, from Eq. (11), that $Z=0$], and 3) if $Y \geq 0$. The first two points depend only on the choice of $Q_1[u]$ and $Q_1[v]$ and are therefore straightforward. The third condition follows from

$$Y = 4(\lambda-\lambda_1)^2[3\alpha_1^2 - 2\alpha_1\alpha_2 + 3\alpha_2^2] - 24\alpha_3 Q_1[u]$$

Substituting for $Q_1[u]$ from Eq. (20) yields

$$Y = 4(\lambda-\lambda_1)^2[3\alpha_1^2 - 14\alpha_1\alpha_2 + 21\alpha_2^2] \geq 0$$

for all values of λ, α_1, and α_2. Therefore, the bifurcation load given by Eq. (20) will always be defined for values of $Q_1[u]$ and $Q_1[v]$ which satisfy 1 and 2.

In the second bifurcation case, Eq. (19) can be expanded to the form

$$-6X\frac{dX}{d\lambda} + \frac{dY}{d\lambda} = \pm 4Y^{1/2}\frac{dX}{d\lambda} \tag{21}$$

Then, squaring both sides and making the appropriate substitutions, the bifurcation load is found as

$$(\lambda-\lambda_1)^2 = \frac{(\alpha_1 - 3\alpha_2)^2\alpha_3 Q_1[u]}{2\alpha_1^2\alpha_2(3\alpha_2 - 2\alpha_1)} \tag{22}$$

This solution exists only if the following conditions are satisfied: 1) the right-hand side of Eq. (22) is positive, 2) $g\{\cdot,\cdot\}$ is positive [see Eq. (15)], and 3) $Y \geq 0$. The first of these depends only on the choice of $Q_1[u]$ and is therefore straightforward. Criterion 2 is a little more complicated. Now

$$Z = [X \pm Y^{1/2}][-3X \pm Y^{1/2}] \tag{23}$$

and from Eq. (21) it follows that

$$\pm Y^{1/2} = \left[-6X\frac{dX}{d\lambda} + \frac{dY}{d\lambda}\right] \Big/ 4\frac{dX}{d\lambda}$$

where it is assumed that $dX/d\lambda \neq 0$. Substitution of $\pm Y^{1/2}$ into Eq. (23) yields, after some algebra,

$$\{Q_1[v]\}^2 = \frac{(\lambda-\lambda_1)^4\alpha_1^2[\alpha_2(2\alpha_1 - 3\alpha_2)]^3}{(\alpha_1 - 3\alpha_2)^4\alpha_3\alpha_4} \cdot 4 \cdot (16)^4$$

which implies that real solutions exist or $g\{\cdot,\cdot\} > 0$ only when

$$\frac{\alpha_2(2\alpha_1 - 3\alpha_2)}{\alpha_3\alpha_4} > 0 \tag{24}$$

The third condition on Y yields, upon substitution for $Q_1[u]$ from Eq. (22),

$$Y = \frac{3(\lambda-\lambda_1)^2}{(\alpha_1 - 3\alpha_2)^2}(\alpha_1 - \alpha_2)^2(\alpha_1 + 3\alpha_2)^2 \geq 0 \tag{25}$$

for all λ, α_1, α_2. A further point of interest is the relationship between $Q_1[u]$ and $Q_1[v]$ which leads to bifurcation. The required result is obtained by eliminating $(\lambda-\lambda_1)$ from Eqs. (11) and (22). Doing so gives

$$\{Q_1[u]\}^2\alpha_2\alpha_3(2\alpha_1 - 3\alpha_2) = \{Q_1[v]\}^2\alpha_1^2\alpha_4 \tag{26}$$

It is important to note that the existence of Eq. (26) relies on the satisfaction of the inequality in Eq. (24).

The secondary bifurcation results can be summarized concisely by substituting the appropriate expressions for α_1, α_2, α_3, and α_4. The two cases are therefore:

Case 1

$$(\lambda-\lambda_1)^2 = \frac{-\{P_{12}[u,v]\}^2 Q_1[u]}{P_2'[v]\{3P_2'[v]P_3[u] - 2P_2'[u]P_{12}[u,v]\}} \tag{27}$$

$$\text{for } Q_1[v] = 0 \tag{28}$$

Case 2

$$(\lambda-\lambda_1)^2 = \frac{\{P_2'[u]P_{12}[u,v] - 3P_2'[v]P_3[u]\}^2 Q_1[u]}{2\{P_2'[u]\}^2 P_2'[v]\{3P_2'[v]P_3[u] - 2P_2'[u]P_{12}[u,v]\}} \tag{29}$$

if

$$P_{12}[u,v]\{3P_2'[v]P_3[u] - 2P_2'[u]P_{12}[u,v]\} > 0 \tag{30}$$

for

$$\{Q_1[u]\}^2 P_2'[v]\{3P_2'[v]P_3[u] - 2P_2'[u]P_{12}[u,v]\} = -\{Q_1[v]\}^2\{P_2'[u]\}^2 P_{12}[u,v] \tag{31}$$

There are several features of these bifurcation results which yield insight into the classification scheme previously mentioned. First, the secondary bifurcation loads defined by Eqs. (27) and (29) cannot occur for the same sign of $Q_1[u]$. That is, if Eq. (27) is defined for $\pm Q_1[u]$, then the only possibility for Eq. (29) is $\mp Q_1[u]$. Also, Eq. (27) has two solutions: one larger and the other less than λ_1. These two secondary bifurcation loads (solutions) lie on different critical load surfaces. If the inequality in Eq. (30) is satisfied, then Eq. (29) has two pairs of solutions, the pairs being defined for a given value of $Q_1[u]$ in conjunction with $\pm Q_1[v]$. The components of a given pair are equal in magnitude and they lie on the same critical load surface. The one pair has a value larger than λ_1, while the other takes a value less than λ_1. Second, if $P_3[u]P_{12}[u,v]$ is positive, the system is classified by Thom[2] as a hyperbolic umbilic catastrophe; and, if it is negative, as an elliptic umbilic catastrophe. The hyperbolic umbilic situation can be further subdivided into the so-called monoclinal and homeoclinal cases, while the elliptic umbilic has been referred to as anticlinal. The terms monoclinal, homeoclinal, and anticlinal have been coined by Thompson and Hunt[12] and refer to the number and slope of possible postbuckling paths for the perfect system.

The final point of interest relates to determination of the subdivisions of the hyperbolic umbilic as well as specification of the primary and complementary critical load-initial imperfection surface. This point is of vital interest as semisymmetric problems yield two critical load surfaces which often intersect each other. Furthermore, both of these surfaces can fold back on themselves with the result that it is

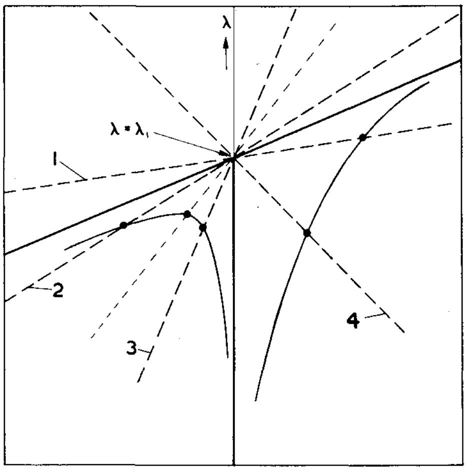

Fig. 1 Schematic representation of load-deflection curves and possible stability boundaries.

possible to have up to six critical loads defined for a given pair of initial imperfection components $Q_I[u]$ and $Q_I[v]$. To make matters worse, for given values of $Q_I[u]$ and $Q_I[v]$ it is not necessary that the least of the possible critical loads be the one which is of engineering significance. The primary surfaces and the subdivisions of the semisymmetric form can be obtained by referring to a figure such as Fig. 1. This figure is a plot of the λ-a plane and demonstrates the prebuckling response of a semisymmetric system if $Q_I[v]$ is set to zero. Thus, by determining the type and magnitude of the critical loads for $Q_I[v]$ equal to zero, it is possible to pinpoint the appropriate critical load-initial imperfection surface for arbitrary $Q_I[u]$ and $Q_I[v]$ under the assumption that the load is applied incrementally starting from zero. The pre- and postbuckling paths of the system, plotted in the λ-a plane, for $Q_I[u]=0$ are given by

$$a=0 \tag{32}$$

and

$$a=\frac{2(\lambda-\lambda_I)P_2'[u]}{3P_3[u]} \tag{33}$$

respectively. These paths are represented by the heavy solid straight lines. The lightly dashed straight line is the stability boundary, which is obtained through a consideration of only mode a deflections. This line is described by the equation

$$a=\frac{(\lambda-\lambda_I)P_2'[u]}{3P_3[u]} \tag{34}$$

A second stability boundary in the λ-a plane comes from a consideration of the b mode deflections. There are four possibilities which are represented by the heavy dashed straight lines and numbered 1 to 4. These lines are represented by

$$a=\frac{(\lambda-\lambda_I)P_2'[v]}{P_{I2}[u,v]} \tag{35}$$

The solid curved lines represent the deflection paths of an imperfect system having an imperfection $Q_I[u]$. The intersection of these curves with the a mode stability boundary will result in a loss of stability at a limit point. On the other hand, if these curves intersect a b mode stability boundary, then the system will lose stability at a point of secondary bifurcation.

Consider the four possibilities which can be obtained from Eq. (35), numbered 1 to 4:

1) This situation arises when $P_3[u]P_{I2}[u,v]>0$, and when the slope of Eq. (35), shown as line 1, is less than the slope of the postbuckling path. That is when

$$|3P_2'[v]P_3[u]|>|2P_2'[u]P_{I2}[u,v]| \tag{36}$$

Thus, since $P_2'[u]$ and $P_2'[v]$ are always negative for initially stable systems, it follows that inequality (30) cannot be satisfied, and only one of the bifurcation possibilities [Eq. (27)] exists. Equation (27) has two solutions, one on each of the primary and complementary surfaces. Also, with reference to Fig. 1, it is noted that the intersection of the imperfect path with line 1 occurs only for a load which is greater than the ideal critical load, λ_I. Thus, the primary critical load surface contains the secondary bifurcation load locus, which is greater than λ_I. This situation is the monoclinal case of the hyperbolic umbilic catastrophe.

2) Case 2 arises when $P_3[u]P_{I2}[u,v]>0$ and when Eq. (35) lies in the position shown as line 2. That is, the slope of 2 is greater than the slope of the postbuckling path but less than the slope of the a mode stability boundary. Therefore,

$$|6P_2'[v]P_3[u]|<|2P_2'[u]P_{I2}[u,v]|>|3P_2'[v]P_3[u]| \tag{37}$$

It follows immediately that inequality (30) is satisfied. There are therefore 1 + 2 loci of bifurcation loads on each of the two surfaces, the one arising from Eq. (27) and a pair from Eq. (28). The 1 + 2 on a given surface are either all greater than λ_I or all less than λ_I. Furthermore, it can be seen that, in the λ-a plane, the only method by which the system can lose stability is at a limit point, that is when the imperfect path strikes the mode a stability boundary. This situation is the homeoclinal case of the hyperbolic umbilic catastrophe. The primary critical load surface is the lower portion of the upper surface, the prescribed portion being contained by the loci of equal pairs of secondary bifurcation loads all of which are greater than λ_I. The reason for the prescribed containment is that the surface folds over on itself at a vertical section which results from the secondary bifurcation phenomenon.

3) The third possibility is that which arises for $P_3[u]P_{I2}[u,v]>0$ and when the slope of Eq. (35) is greater than the slope of the mode a stability boundary. This situation is presented schematically in Fig. 1 as line 3. The requirement on the slopes becomes

$$|P_2'[u]P_{I2}[u,v]|>|3P_2'[v]P_3[u]| \tag{38}$$

which immediately implies that the inequality in Eq. (30) is satisfied. As in the previous case there are 1 + 2 secondary bifurcation load loci on each critical load surface, and the 1 + 2 on a given surface are either all greater than or less than λ_I. Further, from Fig. 1 it can be seen that the primary surface is the one which contains a secondary bifurcation locus which is less than λ_I. This situation is the second possibility of the homeoclinal case of the hyperbolic umbilic catastrophe. The primary surface is the bottom portion of the lower critical load surface which is contained by the vertical slopes. As in the previous case the vertical slopes occur when the surface folds over on itself.

4) The final possibility arises when $P_3[u]P_{I2}[u,v]<0$ which implies that the slope of Eq. (35) is of opposite sign to that of both the postbuckling path and the mode a stability boundary. Inequality (30) is always satisfied which again implies the existence of 1 + 2 secondary bifurcation loci on each of the critical load surfaces. The 1 + 2 loci are as usual either all greater or all less than λ_I on a prescribed critical load surface. Figure 1 indicates that the primary critical load surface is the one for which one of the bifurcation loci is less than λ_I. As the case described is the elliptic umbilic

catastrophe or the anticlinal case, this means that the lower surface is the one of interest. A further feature is that the critical load surface contains discontinuities along lines which correspond to the secondary bifurcation loci of Eq. (28). Along these lines there is a vertical slope and then a jump to another surface. This phenomenon may be observed in the figures presented at the end of the paper.

These four cases are summarized in Table 1. This table represents an extension of a table by Thompson and Hunt[12] and provides all of the information necessary to isolate the portion of the critical load-initial imperfection surface which is relevant in an engineering context.

Evaluation of Critical Load-Initial Imperfect Surfaces for a Beam on a Nonlinear Elastic Foundation

The results of the previous section are completely general; however, before it is possible to evaluate the critical load-initial imperfection surfaces it is necessary to assign values to the energy functionals. Also, rather than choosing these values arbitrarily, it seems best to present an example and thereby give more insight into the whole procedure.

The example chosen is that of an axially loaded beam which is resting on a nonlinear elastic foundation. Besides being relatively easy to treat, this problem is very informative as all possible cases arising from the semisymmetric form can be obtained through a variation of the foundation spring stiffness. The monoclinal case of the hyperbolic umbilic for this particular example has been treated previously in Ref. 18.

The potential energy functionals for the beam problem are

$$P_1^\lambda[\psi]=0 \tag{39a}$$

$$P_2^\lambda[\psi]=\int_0^1 [(\psi'')^2-\lambda(\psi')^2+K_1\psi^2]\,d\xi \tag{39b}$$

$$P_3^\lambda[\psi]=\int_0^1 \left[2K_2\frac{\psi^3}{3}\right]d\xi \tag{39c}$$

$$\vdots$$

$$Q_1^\lambda[\psi]=-2\lambda\int_0^1 [\psi_0'\psi']\,d\xi \tag{39d}$$

where $\psi=\psi(\xi)$ is the deflection, ξ is the axial coordinate, K_1, K_2 are the linear and quadratic spring constants, respectively, $\psi_0=\psi_0(\xi)$ is the initial imperfection, and the beam is assumed to be of unit length. The ideal critical load or the buckling load of the system is obtained as the least eigenvalue of the linearized problem. It is

$$\lambda_1=\frac{(m\pi)^4+K_1}{(m\pi)^2} \tag{40}$$

with the corresponding eigenfunction

$$\psi_1=a\sin m\pi\xi \tag{41}$$

where m is the buckling mode number. The appropriate value of m is obtained through the requirement that m be the integer that minimizes λ_1 for a prescribed value of K_1. A further investigation reveals that when

$$K_1=(m+1)^2m^2\pi^2 \tag{42}$$

then, λ_1 has two equal minima for the buckling mode numbers m and $m+1$. This is the particular combination of circumstances which leads to the multiple-mode situation. The eigenvalue and eigenfunction for the multiple-mode situation are then, respectively,

$$\lambda_1=(m\pi)^2+(m+1)^2\pi^2 \tag{43}$$

and

$$\psi_1=a\sin m\pi\xi+b\sin(m+1)\pi\xi \tag{44}$$

where a and b are the modal amplitudes. Evaluating the required energy functionals yields

$$P_2'[u]=-\tfrac{1}{2}(m\pi)^2 \tag{45a}$$

$$P_{11}'[u,v]=0 \tag{45b}$$

$$P_2'[v]=-\tfrac{1}{2}(m+1)^2\pi^2 \tag{45c}$$

$$P_3[u]=\frac{8\beta_1}{9(m\pi)}\frac{1}{2}(1-(-1)^m)+\frac{\beta_2}{4} \tag{45d}$$

$$P_{21}[u,v]=\frac{8\beta_1}{(m+1)\pi}\left\{\frac{m^2}{4m^2-(m+1)^2}\right\}\times\tfrac{1}{2}(1-(-1)^{m+1})+\beta_3 \tag{45e}$$

$$P_{12}[u,v]=\frac{8\beta_1}{(m\pi)}\left\{\frac{(m+1)^2}{4(m+1)^2-m^2}\right\}\times\tfrac{1}{2}(1-(-1)^m)+\beta_2 \tag{45f}$$

Table 1 Summary of the semisymmetric form for determination of the primary critical load-initial imperfection surface

Category (Thom)	Subdivision (Thompson and Hunt)	Number of secondary bifurcations	Primary critical load surface
	1) $\lvert 3P_3[u]P_2'[v]\rvert > \lvert 2P_{12}[u,v]P_2'[u]\rvert$ (Monoclinal)	1 locus on each surface	Secondary bifurcation load greater than λ_1
$P_3[u]P_{12}[u,v]>0$ (Hyperbolic umbilic)	2) $\lvert 6P_3[u]P_2'[v]\rvert > \lvert 2P_{12}[u,v]P_2'[u]\rvert > \lvert 3P_3[u]P_2'[v]\rvert$ (Homeoclinal)	1+2 loci on each surface	All secondary bifurcation loads greater than λ_1
	3) $\lvert 3P_3[u]P_2'[v]\rvert < \lvert P_{12}[u,v]P_2'[u]\rvert$ (Homeoclinal)	1+2 loci on each surface	All secondary bifurcation loads less than λ_1
$P_3[u]P_{12}[u,v]<0$ (Elliptic umbilic)	4) None (Anticlinal)	1+2 loci on each surface	Lower surface

$$P_3[v] = \frac{8\beta_1}{9(m+1)\pi}\,\frac{1}{2}\,(1-(-1)^{m+1}) + \frac{\beta_3}{4} \tag{45g}$$

$$Q_1[u] = -2\lambda_1(m\pi)^2\int_0^1 \psi_0(\xi)\,\sin m\pi\xi\;d\xi \tag{45h}$$

$$Q_1[v] = -2\lambda_1(m+1)^2\pi^2\int_0^1 \psi_0(\xi)\,\sin(m+1)\pi\xi\;d\xi \tag{45i}$$

where the eigenfunction (44) has been taken in the form $\psi_1 = au + bv$, and the quadratic spring constant has been chosen as

$$K_2 = \beta_1 + \beta_2\,\sin m\pi\xi + \beta_3\,\sin(m+1)\pi\xi \tag{46}$$

Figures 2-5 are projections of critical load-initial imperfection surfaces for all possible cases of the semisymmetric form. In these figures, the imperfections are given in terms of the nondimensional imperfection parameters Y_1 and Y_2, which are defined by

$$Y_1 = \frac{3\bar{P}_3[u]}{\{\lambda_1 P_2'[u]\}^2}\,Q_1[u]$$

and

$$Y_2 = \frac{3\bar{P}_3[v]}{\{\lambda_1 P_2'[v]\}^2}\,Q_1[v]$$

where $\bar{P}_3[u] = 20K_1/27m\pi$, and $\bar{P}_3[v] = 20K_1/[27(m+1)\pi]$. Only that portion of the surface which is of engineering relevance has been retained, and two projections, one onto each of the λ-Y_1 and λ-Y_2 planes, have been considered for each case. The dashed curves in Figs. 3-5 indicate secondary bifurcation situations with a vertical slope on the critical load-initial imperfection surface. These dashed curves are defined for combinations of $Q_1[u]$ and $Q_1[v]$ which satisfy Eq. (31). The light solid line in Fig. 5 is a locus of cusps for various combinations of $Q_1[u]$ and $Q_1[v]$. It should be noted that these examples are naturally semisymmetric by virtue of the energy functionals. That is, $P_{21}[u,v]$ and $P_3[v]$ can be made to vanish when m is odd, while $P_3[u]$ and $P_{12}[u,v]$ can vanish when m is even. In Figs. 2-4, $\beta_1 = 5K_1/6$, and $\beta_2 = \beta_3 = 0$. A surface of the form found in Fig. 2 arises when $m = 1$ or 3 although the specific result is given for $m = 1$. This figure is the monoclinal form of the hyperbolic umbilic catastrophe. Figure 3 results when $m \geq 5$ and is odd. It is the upper surface of the homeoclinal form of the hyperbolic umbilic. The figure is evaluated for $m = 5$. The lower surface of the homeoclinal form of the hyperbolic umbilic is shown in Fig. 4. This situation occurs for m even, and the present result has been obtained for $m = 2$.

The generation of the critical load-imperfection surfaces for the anticlinal case or the elliptic umbilic catastrophe requires a little more care. For the present example, it occurs when $\beta_2 \neq 0$, $\beta_3 \neq 0$, and m is even and with the appropriate choices for β_2 or β_3 in either of the two cases. The curves shown in Fig. 5 are based on values of $m = 1$, $\beta_2 = 8\beta_1/3\pi$, and $\beta_1 = 20K_1/3$.

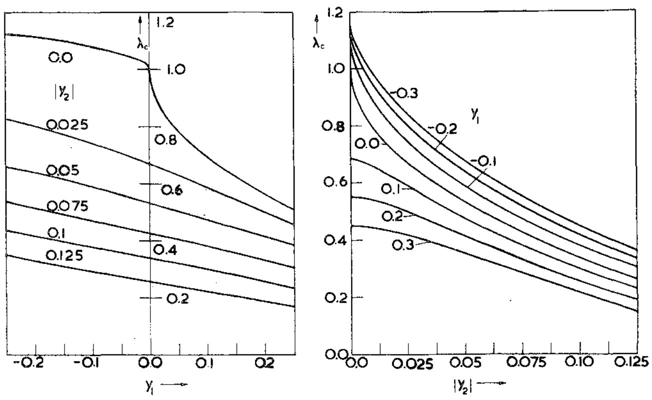

Fig. 2 Hyperbolic umbilic: Cross sections through the critical load (λ_c)—initial imperfection (y_1, y_2) surfaces: $|3P_3[u]P_2'[v]| > |2P_{12}[u,v]P_2'[u]|$.

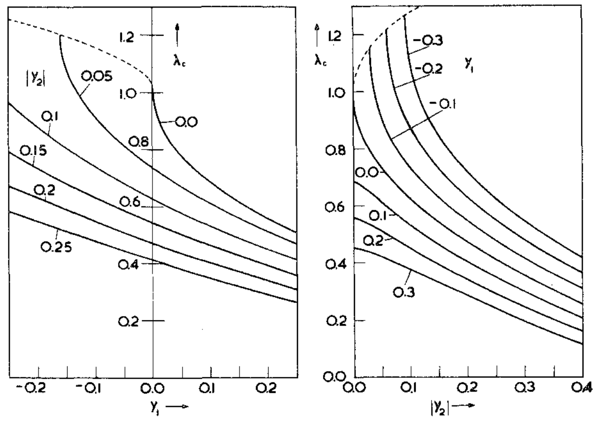

Fig. 3 Hyperbolic umbilic: Cross sections through the critical load (λ_c)—initial imperfection (y_1, y_2) surfaces: $|6P_3[u]P_2'[v]| > |2P_{12}[u,v]P_2'[u]| > |3P_3[u]P_2'[v]|$.

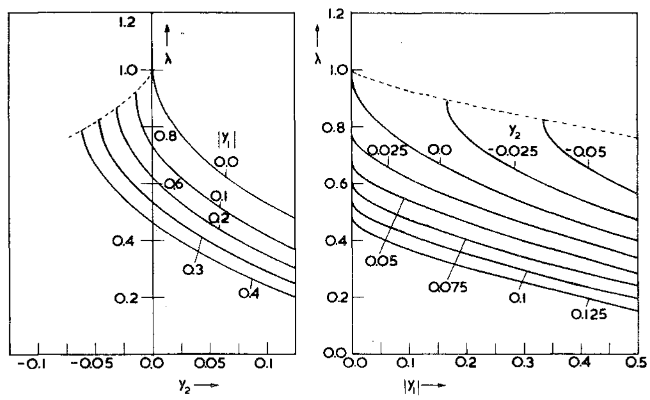

Fig. 4 Hyperbolic umbilic: Cross sections through the critical load (λ_c)—initial imperfection (y_1, y_2) surfaces: $|3P_3[u]P_2'[v]| < |P_{12}[u,v]P_2'||$.

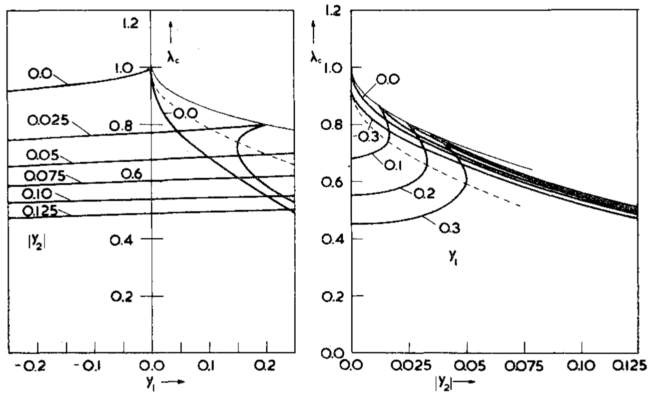

Fig. 5 Elliptic umbilic: Cross sections through the critical load (λ_c)—initial imperfection (y_1, y_2) surfaces.

Summary

The present article has presented an asymptotic analysis of two-mode buckling problems when the highest-order terms contained in the potential energy are of the so-called semisymmetric form. This potential energy expression has been investigated in detail and an asymptotic analysis has yielded closed-form solutions for the three-dimensional critical load-initial imperfection surfaces. All secondary bifurcation cases which result from an interaction of the two principal modes have been isolated and closed-form solutions for the bifurcation loads have also been obtained.

The results are summarized in Table 1. This table indicates the class of surface as well as the portion of the surface which will be of primary importance in a stability analysis. The primary surface is considered in the sense that it represents the surface which will be intersected first as a system is loaded from an initial unloaded (stable) configuration. Thus, any problem which reduces to the semisymmetric form of Eq. (7) can be solved based on the present results. These results are complemented by the example of an axially loaded beam resting on a nonlinear elastic foundation. This example is particularly interesting as it can be manipulated to demonstrate all possible semisymmetric cases.

References

[1] Hutchinson, J.W. and Koiter, W.T., "Postbuckling Theory," *Applied Mechanics Reviews,* Vol. 23, Dec. 1970, pp. 1353-1366.

[2] Thompson, J.M.T. and Hunt, G.W., *A General Theory of Elastic Stability,* Wiley, New York, 1973.

[3] Huseyin, K., *Nonlinear Theory of Elastic Stability,* Noordhoff, Leyden, 1974.

[4] Budiansky, B. (Ed.), "Buckling of Structures," *IUTAM Symposium,* Cambridge, June 17-21, 1974.

[5] Supple, W.J., "Coupled Branching Configurations in the Elastic Buckling of Symmetric Structural Systems, *International Journal of Mechanical Sciences,* Vol. 9, Jan. 1967, pp. 97-112.

[6] Chilver, A.H., "Coupled Modes of Elastic Buckling," *Journal of the Mechanics and Physics of Solids,* Vol. 15, Jan. 1967, pp. 15-28.

[7] Sewell, M.J., "On the Branching of Equilibrium Paths," *Proceedings of the Royal Society, London,* Vol. A315, March 1970, pp. 499-518.

[8] Johns, K.C. and Chilver, A.H., "Multiple Path Generation at Coincident Branching Points," *International Journal of Mechanical Sciences,* Vol. 13, Nov. 1971, pp. 899-910.

[9] Ho, D., "The Influence of Imperfections on Systems with Coincident Buckling Loads," *International Journal of Nonlinear Mechanics,* Vol. 7, June 1972, pp. 311-321.

[10] Ho, D., "Buckling Load of Nonlinear Systems with Multiple Eigenvalues," *International Journal of Solids and Structures,* Vol. 10, Nov. 1974, pp. 1315-1330.

[11] Thom, R., *Structural Stability and Morphogenesis,* Benjamin, Reading, 1975.

[12] Thompson, J.M.T. and Hunt, G.W., "Towards a Unified Bifurcation Theory," *Journal of Applied Mathematics and Physics,* Vol. 26, Sept. 1975, pp. 581-604.

[13] Hansen, J.S., "On the Relationship Between Some Potential Energy Expressions and the Elementary Catastrophes," to be published.

[14] Koiter, W.T., "On the Stability of Elastic Equilibrium," Dissertation, Polytechnic Institute, Delft, Holland, 1945.

[15] Tvergaard, V., "Imperfection-Sensitivity of a Wide Integrally Stiffened Panel Under Compression," *International Journal of Solids and Structures,* Vol. 9, Jan. 1973, pp. 177-192.

[16] Hutchinson, J.W., "Imperfection Sensitivity of Externally Pressurized Spherical Shells," *Journal of Applied Mechanics, Transactions of ASME,* Ser. E, Vol. 34, March 1967, pp. 49-55.

[17] Keener, J.P., "Perturbed Bifurcation Theory at Multiple Eigenvalues," *Archive for Rational Mechanics and Analysis,* Vol. 56, No. 4, 1975, pp. 348-366.

[18] Hansen, J.S., "Buckling of Imperfection Sensitive Structures: A Probabilistic Approach," Dissertation, Univ. of Waterloo, Waterloo, Canada, 1973, Chap. 7.